$x_1=3, 5, 7, 10;\quad x_2=2, 4, 7, 8, 9, 10;\quad x_3=2, 5, 6, 7, 8;$
$x_4=4, 5, 6, 7, 10;\quad x_5=2, 3, 4, 5, 8, 10;\quad x_6=2, 5, 7, 9.$

For modulo 17, $35x_i \equiv x_i$. Hence

$x_1=3, 6, 8, 9;\quad x_2=2, 5, 7, 8;\quad x_3=2, 3, 6, 7, 8;$
$x_4=2, 3, 5, 8, 10;\quad x_5=3, 5, 6, 9, 10;\quad x_6=2, 4, 5, 8, 9, 10.$

The values of the x_i common to the three sets are

$$x_1=3;\quad x_2=2, 8;\quad x_3=7;\quad x_4=10;\quad x_5=5;\quad x_6=5, 9.$$

Modulus 19 excludes $x_2=2$, $k\equiv 16$; $x_5=5$, $k\equiv 7$; $x_6=5$, $k\equiv 8$. Modulus 23 excludes $x_2=8$, $k\equiv 7$; $x_3=7$, $k\equiv 0$; $x_6=9$, $k\equiv -7$. Of the two remaining values,* $x_1=3$ gives $k=107$, $l=6k+2=644$, whence

$$Q=(13799)^2,\quad n_1-n=13799,\quad n_1+n=16745,\quad n_1=15272,$$
$$n=1473,\quad 1+26n=38299,\quad 1+26n_1=397073,$$

the two prime factors of a.

3. We next determine the composition of

$$b=31401724537=\frac{56^7-1}{56-1}=1+56N,\quad N\equiv\frac{56^6-1}{56-1}.$$

By Fermat's theorem, a prime factor p of b has the form $14k+1$ and hence $56l+1$, $+15$, $+29$, $+43$. The second and third forms are excluded by Legendre's table of divisors of quadratic forms, or as follows. If $p=56l+15$, 2 is a quadratic residue of p, and 7 a non-residue, in view of the reciprocity law. Thus 14 is a non-residue of p, contrary to $56^8\equiv 56 \pmod{p}$.

If $b=pq$, then $q\equiv 1 \pmod{14}$, $q=1+14k_1$. Thus

$$k+k_1+14kk_1=4N=4.57(56^4+56^2+1).$$

Hence $k+k_1\equiv 4 \pmod{14}$, so that

$$k+k_1=4+14h,\quad kk_1+h=16.57(56^3+56)+16. \tag{3}$$

First let $k=4l+3$, so that $p=56l+43$. By (3), $k_1+3\equiv 2h$, $3k_1+h\equiv 0 \pmod 4$, whence $k_1\equiv h\equiv 3 \pmod 4$. Set $h=4t+3$, $k_1=4l_1+3$. Then by (3),

*For $x_4=10$, $k=367\equiv -10 \pmod{29}$, $Q\equiv 3$, a quadratic non-residue of 29.

$$l+l_1=14t+10, \quad 4ll_1+43t=4.57(56^3+56)-29.$$

By the latter, $t\equiv 1 \pmod 4$, $t=4c+1$. Hence

$$\tfrac{1}{4}(l_1-l)^2=S=16(7c+3)^2+43c+18-57(56^3+56).$$

Thus $S\equiv 3c+2 \pmod 8$. If c is odd, $S\equiv 1$, $c\equiv 5 \pmod 8$. If c is even, S must be a multiple of 4, whence $c\equiv 2 \pmod 4$, $c=4m+2$, $\frac{1}{4}S\equiv 43m \pmod{32}$. If m is odd, then $m\equiv 3 \pmod 8$. If m is even, then $m=4r$, and $\frac{1}{16}S\equiv 3r \pmod 8$, so that either r is a multiple of 4 or $r\equiv 3 \pmod 8$. Hence we have the cases

$$(4) \qquad c\equiv 5 \pmod 8, \quad c=32w+14, \quad 64w+2, \quad 128w+50.$$

Modulo 81, S is the product of 16 by $S'=49c^2+70c+15$. In particular, $S'\equiv c^2+c \pmod 3$. Thus $c\equiv 0$ or $2 \pmod 3$. If c is a multiple of 3, S' must be a multiple of 9, so that $c=3+9d$, $S'\equiv 9(4d+2)$, mod. 81. Thus, $4d+2\equiv 0, 1, 4, 7 \pmod 9$, $d\equiv 4, 2, 5, 8$. Next, if $c=2+3e$, $S'\equiv 6e \pmod 9$, $e=3f$. Thus $S'\equiv 9(5f+3)$, mod. 81, $f\equiv 2, 3, 5, 8 \pmod 9$. Hence

$$(5) \qquad c\equiv 20, 21, 29, 39, 47, 48, 74, 75 \pmod{81}.$$

$S\equiv 4c^2+3$, $c\equiv \pm 2 \pmod 5$; $S\equiv c+1$, $c\equiv 0, 1, 3, 6 \pmod 7$;
$S\equiv 16(5c^2+3)$, $c\equiv 0, \pm 2, \pm 3 \pmod{11}$;
$S\equiv 4(c^2+1)$, $c\equiv 0, \pm 3, \pm 4, \pm 5 \pmod{13}$;
$S\equiv 2[(c-4)^2-2]$, $c\equiv 2, 3, 4, 5, 6, 10, 11, 14, 15 \pmod{17}$.

By the tables cited, b has no factor $<10^4$. Thus, $1+14k\geq 9999$, $1+14k_1\leq 3140172$. Hence $k\geq 714$, $k_1<224298$, $l\geq 178$, $l_1<56074$. The sum of the latter gives the maximum $l+l_1$. Thus $t<4018$, $c\leq 1004$. Since $\sqrt{b}$ just exceeds 177205, the approximate value for equal k's just exceeds 12657. Thus the equal l's just exceed 3164. Hence $t\geq 451$, $c>112$.

For the first case under (4), we set $c=81x+20, \ldots, 75$, by (5). Thus

$$5\equiv x+4, 5, 5, 7, 7, 0, 2, 3; \quad x\equiv 1, 0, 0, 6, 6, 5, 3. 2 \pmod 8.$$

The resulting values of c between 112 and 1005 are

525, 669, 749, 885, 965; 533, 237, 677; 453, 317.

The first five are excluded by mod. 5, the next three by mod. 11, the last two by mod. 7.

For $c=32w+14$, $4 \leq w \leq 30$ by the limits on c. By (5),

$$w \equiv 66, 23, 3, 59, 39, 77, 12, 50 \pmod{81},$$

respectively. Hence $w=23, 12$. But 23 is excluded mod. 5, and 12 mod. 17.

For $c=64w+2$, $2 \leq w \leq 15$. But $w \equiv 0, 4 \pmod 5$, $w \equiv 0, 1, 2, 5, 8 \pmod{11}$. Hence $w=5$, $c=322 \equiv 79 \pmod{81}$, and is excluded by (5).

For $c=128w+50$, $1 \leq w \leq 7$. By (5), $w=3$, $c=434$, excluded mod. 5.

4. It remains to determine whether or not b has a factor $1+56n$. The complementary factor is of the form $1+56n_1$. Hence

$$n+n_1+56nn_1=N.$$

By inspection, $N \equiv 1 \pmod{56}$. Hence there is an integer l for which

$$n+n_1=56l+1, \quad nn_1+l=C=\frac{56^5-1}{56-1}=10013305,$$
$$(n_1-n)^2=S=(56l+1)^2+4l-4C.$$

Modulo 56, $S \equiv 4l-3$. Thus $S \equiv 1 \pmod 8$, $l \equiv 1 \pmod 2$, $l=2\lambda+1$. Also $S \equiv \lambda+1 \pmod 7$, $\lambda \equiv 0, 1, 3, 6 \pmod 7$. We have

$$S=112^2\lambda^2+8.1597\lambda-40049967.$$

Modulo 81, S is the product of $112^2 \equiv -11$ by $\sigma=\lambda^2+2\lambda+15$. The latter must be a quadratic residue of 81. In particular, $\lambda+1 \equiv \pm 1+3t$. Then $\sigma \equiv 6 \pm 6t \pmod 9$; thus $\sigma \equiv 0 \pmod 9$, $t \equiv \mp 1 \pmod 3$, $\lambda+1=\mp 2+9f$. Then $\sigma \equiv 9(2 \mp 4f)$, mod. 81. Thus $2 \mp 4f$ is one of the quadratic residues 0, 1, 4, 7 of 9, whence $\pm f \equiv 5, 7, 4, 1 \pmod 9$. Hence

$$\lambda \equiv 6, 19, 33, 37, 42, 46, 60, 73 \pmod{81}.$$
$$4S \equiv (\lambda+2)^2-2, \ \lambda \equiv 2, 4 \pmod 5;$$
$$S \equiv 4[(\lambda+2)^2+4], \ \lambda \equiv 2, 5, 8, 9, 10 \pmod{11};$$
$$S \equiv 25[(\lambda-5)^2-3], \ \lambda \equiv 0, 1, 3, 5, 7, 9, 10 \pmod{13};$$
$$8S \equiv (\lambda+2)^2-9, \ \lambda \equiv \pm 1, -2, \pm 3, -5, 6, \pm 7 \pmod{17}.$$

Since b has no factor $<10^4$, $n>178$, $n_1<56075$. The maximum $n+n_1$ is approximately 56253, whence $l<1005$, $\lambda<502$. The minimum $n+n_1$ is given by $n=n_1=3165-$. Hence $l \geq 113$, $\lambda \geq 56$. From the above residues moduli 81 and 5,

$$\lambda=405t+19, 37, 42, 87, 114, 127, 154, 199, 204, 222, 249, 262, 289, 357, 384, 397.$$

For the first three $t=1$; for the fourth $t=0, 1$; for the others $t=0$. Of the 17 resulting values of λ, 114, 222, 249, 289, 397, 424, 492 are excluded by mod. 7; then 127, 154, 199, 204, 447 are excluded by mod. 11; 262 and 357 by mod. 13; 87 and 442 by mod. 17; for the remaining value $\lambda=384$, $S\equiv 21$ (mod. 23), whereas 21 is a quadratic non-residue of 23.

Hence $b=\frac{1}{5}(56^7-1)$ *is a prime.*

While it is believed that the above work is accurate, having been carefully checked, it should be added that the same result was found by an earlier proof different as to details.

5. By the same method, I obtain the following results:

$$56^7+1=3.19.15737.1925393,$$
$$34^{17}+1=5.7.307.443.1531.28051.112643.4708729,$$
$$52^{13}+1=53.4057.21841.4328028093013,$$

all of the given factors being prime. That the last number of 13 digits is prime, I have verified by two proofs differing as to details. The factor 21841 was found by accident by Lt. Col. Cunningham. I ran across the factor 112643 of the second number in the manner explained in the *Quarterly Journal*, 1908, page 45; but the remaining two large factors were found by the present method.

6. In view of the interest in the numbers m^m-1 and their importance in connection with the last theorem of Fermat, it is desirable that some arithmetician should check the statement of E. Lucas (*American Journal of Mathematics*, Vol. 1, 1878, p. 294) that the large factors of 10 and 12 digits in $22^{11}\pm 1$ are actually primes. For a verification by the present method it is of the greatest help to know that there are no factors less than 10,000, in view of the tables by Lt. Col. Cunningham. The latter believes that Lucas intended to record his factors as primes; but that an uncertainty runs right through his factorizations as to the primality of the factors, no clue whatever being given as to how the primality was detected.

FACTORING IN A DOMAIN OF RATIONALITY.

By ELIZABETH R. BENNETT, The University of Illinois.

If a series of symbols $R_1, R_2, \ldots$ which are supposed to obey the ordinary laws of algebra, but are not necessarily thought of as representing numbers, are combined with respect to the four fundamental operations of arithmetic—addition, subtraction, multiplication, and division, division by zero being excluded, there result a series of expressions which are rational

with respect to these symbols. The totality of such expressions is called a domain of rationality and it is the smallest possible domain involving the symbols $R_1, R_2, \ldots$

If only one of these symbols as R_1 is involved in the combinations and if R_1 is a rational number different from zero, the domain of rational numbers including zero, positive and negative integers, and positive and negative rational fractions is obtained. This domain of rational numbers is included in every domain. The complex numbers of form $a+bi$, where a and b are rational numbers and $i=\sqrt{-1}$, also constitute a domain of rationality which includes the domain of rational numbers.

When we add or adjoin to a known domain a number β which does not already belong to it, the new set of numbers constitutes a domain, if we also add to it all numbers obtained from additions, subtractions, multiplications, and divisions involving β and all numbers of the original domain. The domain of the ordinary complex numbers already mentioned may be formed by the adjunction of i to the domain of rational numbers.

An algebraic integer is a root of the equation

$$x^n+a_1x^{n-1}+a_2x^{n-2}\ldots+a_n=0$$

where $a_1, a_2, \ldots, a_n$ are rational integers. An algebraic quadratic integer is a root of the above equation when $n=2$. All algebraic quadratic integers are of the form $x+y\sqrt{m}$ when $m\equiv 2$ or $3 \pmod 4$, and of the form $x+y\frac{1+\sqrt{m}}{2}$ when $m\equiv 1 \pmod 4$. It is assumed that m is not divisible by any square greater than 1 and that x and y are integral.

The term integral domain is understood, as usual, to mean a set of integral elements which is invariant with respect to addition, subtraction, and multiplication; that is, any combinations of the numbers of the set by the operations mentioned yield again a number belonging to the set. An integral algebraic domain is then an integral domain formed by the adjunction of an algebraic integer to the ordinary integral domain.

In the ordinary integral domain the theorem that a number can be resolved into its positive prime factors in only one way is fundamental. That the above theorem is not valid in all domains is well known and may be easily shown by an example. For instance, $6=2\times 3=(1+\sqrt{-5})(1-\sqrt{-5})$, the factors of each product being primes in the domain considered. It was in order to avoid such possibilities in factoring, as indicated by the above example, that the theory of ideals was created.

It is also generally known that numbers prime in one domain may be composite in another. For example, in the ordinary integral complex domain every rational prime of the form $4n+3$ is a complex prime and every rational prime of the form $4n+1$ is composite.

In the *Nouvelles Annales de Mathematiques* for 1903, M. G. Fontené

has considered algebraic integers of the form $x+y\sqrt{-5}$. In this article it is shown that factoring in the domain of the numbers $x+y\sqrt{-5}$ is not unique, but if the domain is enlarged so as to contain numbers of the form $\frac{x+y\sqrt{-5}}{\sqrt{2}}$ factoring becomes a unique process.

The relation which exists between factoring in a quadratic complex domain and the number of classes of quadratic forms corresponding to the domain has been considered by A. E. Westun in "Certain Systems of Quadratic Complex Numbers," *Transactions of the Cambridge Philosophical Society*, Vol. XVII, 1899. In both of the articles mentioned, factoring is not restricted to operations in an integral quadratic domain.

From the previous definitions and illustrations which are quite generally known, it is evident that an integral algebraic domain can be found by the adjunction of $\sqrt{-6}$ to the ordinary integral domain. This integral algebraic domain will be denoted for brevity by the letter Ω, and some of the properties of numbers in the domain will be considered with special reference to factoring.

We now prove the following theorem:

THEOREM I. *All ordinary primes of the form* $24z+1$ *and* $24z+7$ *are composite in* Ω.

From the theory for binary quadratic forms, it is known that D, the determinant of the form, must be a quadratic residue of m, where m is the number to be represented by the form. The number -6 is a quadratic residue of primes of the form $24z+1$, $24z+7$, $24z+5$, and $24z+11$. There are only two reduced forms for $D=-6$, namely, x^2+6y^2 and $2x^2+3y^2$. The form $2x^2+3y^2$ will not give complex factors in Ω, and so need not be considered. Then all ordinary primes which are composite in Ω must be represented by x^2+6y^2.

If m is the prime to be represented, $m=x^2+6y^2$ and, therefore, $x^2\equiv m \pmod 6$. $x^2\equiv 1 \pmod 6$, but x^2 is not congruent to 5 (mod. 6) and, therefore, all ordinary primes represented by x^2+6y^2 are of form $24z+1$ and $24z+7$ and only primes of these forms can be factored in Ω.

$$(1-\sqrt{-6})(1+\sqrt{-6})=7; \quad (5-\sqrt{-6})(5+\sqrt{-6})=31.$$

THEOREM II. *Primes of the form* $24z+1$ and $24z+7$ *can be resolved into their complex prime factors in only one way in* Ω.

There are four representations of x^2+6y^2 which give the same prime m. There are two solutions of the congruence $n^2\equiv -6 \pmod m$, and two substitutions transforming x^2+6y^2 into an equivalent form. Therefore, a prime number can be represented by x^2+6y^2 in only one way and consequently primes of the form $24z+1$ and $24z+7$ can be resolved into prime complex factors in only one way in Ω.

THEOREM III. *In order that a composite rational integer may be resolved into complex factors only in* Ω, *it must be of the form,* $m=a^{(\alpha)}b^{(\beta)}c^{\gamma}d^{\delta}$, *where a represents primes of the form* $24z+1$ *and* $24z+7$, *and* (α) *the number of primes in a; b represents primes of the form* $24z+5$ *and* $24z+11$ *and* (β) *the number of primes in b;* $c=2$, $d=3$, $(\beta)+\gamma+\delta$ *is an even number and* $\gamma+\delta \geqslant (\beta)$.

The determinant of this form, that is, -6, must be a quadratic residue of any composite number m properly represented by x^2+6y^2 and, therefore, must be a quadratic residue of every prime factor of m. $x^2+6y^2=m$, or m must be a quadratic residue of 6. The number -6 is a quadratic residue of primes of the form $24z+1$, $24z+7$, $24z+5$, and $24z+11$. Primes of form $24z+1$ and $24z+7$ are quadratic residues of 6, while primes of the form $24z+5$ and $24z+11$ are non-quadratic resides of 6. Then any number $m=a^{(\alpha)}b^{(\beta)}$ is properly representable by x^2+6y^2, or may be resolved into complex factors only in Ω, (β) being even, since an even number of non-quadratic residues is a quadratic residue. We have the following equations:

$$\text{(I)}. \quad 2x^2+3y^2=(x\sqrt{2}+y\sqrt{-3})(x\sqrt{2}-y\sqrt{-3}).$$

$$\text{(II)}. \quad \text{Then } 2(2x^2+3y^2)=\sqrt{2}(x\sqrt{2}+y\sqrt{-3}).\sqrt{2}(x\sqrt{2}-y\sqrt{-3})$$
$$=(2x+y\sqrt{-6})(2x-y\sqrt{-6})=(z+y\sqrt{-6})(z-y\sqrt{-6}).$$

$$\text{(III)}. \quad \text{Also, } 3(2x^2+3y^2)=\sqrt{3}(y\sqrt{3}+x\sqrt{-2}).\sqrt{3}(y\sqrt{3}-x\sqrt{-2})$$
$$=(3y+x\sqrt{-6})(3y-x\sqrt{-6})=(z'+x\sqrt{-6})(z'-x\sqrt{-6}).$$

The primes 2 and 3 are not represented by x^2+6y^2 and are not properly represented by $2x^2+3y^2$. Equations (II) and (III) then show that neither 2^{γ}, 3^{δ}, nor $2^{\gamma}3^{\delta}$ are properly represented, or have complex factors in the domain Ω. Primes of the form $24z+5$ and $24z+11$, however, are properly represented by $2x^2+3y^2$ and an inspection of equations (II) and (III) will show that the product of any such prime b by either 2 or 3 will give a number having complex factors in Ω. Since neither 2^{γ}, 3^{δ} nor $2^{\gamma}3^{\delta}$ have complex factors in Ω, but $2b$ and $3b$ have such factors, $\gamma+\delta$ cannot be greater than (β), $(\beta)-\gamma+\delta$ must be even since all primes of form $24z+5$ and $24z+11$ are non-quadratic residues, (mod. 6). Therefore, $(\beta)+\gamma+\delta$ must be an even number.

THEOREM IV. *Any composite rational integer* $m=a^{(\alpha)}b^{(\beta)}c^{\gamma}d^{\delta}$ *representable by the form* x^2+6y^2 *can be resolved into its prime complex factors in more than one way provided it contains at least two different prime factors of the form* $24z+5$ and $24z+11$, *so that* $(\beta)-(\gamma+\delta)<\alpha$.

Let Kx represent the composite rational integer, x representing the product of prime factors of form $24z+5$ and $24z+11$, K the product of all other prime factors. K can be resolved into its complex prime factors in only one way because primes of form $24z+1$ and $24z+7$ are resolvable into

complex factors in only one way and 2^γ, 3^δ, or $2^\gamma 3^\delta$ have no complex factors in the domain. It is also evident from the previous theorems that no prime of the form $24z+1$ and $24z+7$ combined with either 2 or 3 or with a single prime of the form $24z+5$ or $24z+11$ can be resolved into complex factors only in Ω. $2b$ and $3b$, where b is any prime of form $24z+5$ and $24z+11$, can be resolved into complex factors in only one way since 2 and 3 have no complex factors in Ω. From these statements, it is clear that the factoring in different distinct ways must depend only on the factors of x.

A number x can be represented or resolved into its complex factors in Ω in 2^{w-1} ways where w represents the number of different prime factors of x. These $w-1$ representations will be distinct, since primes of form $24z+5$ and $24z+11$ are also primes in the domain Ω.

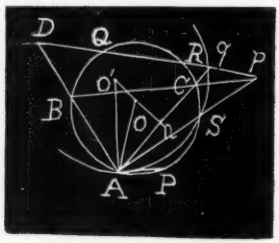
NOTE ON THE STEINER POINT.

By W. GALLATLY, Swanage, England.

Let ABC be the mid-point triangle, and PQR the pedal triangle of a given triangle $A'B'C'$, so that AP is parallel to BC, and $AQ=AR=BC=a$: $(a>b>c)$. And since rq is antiparallel to BC, $\angle Arq=C$, $\angle Aqr=B$.

Describe a circle around $\triangle Aqr$. This touches AP since $\angle Arq=C=\angle CAP$, and therefore the center O' lies on the perpendicular from A on BC. Join Ap, cutting the circle ABC in S. Then S is the Steiner point of the triangle ABC.

Since q, C, B, r are cyclic, $pq.pr=pC.pB=pS.pA$. Hence S lies on the circle Aqr. To find the radius, ρ, of this circle, we have $Aq=2\rho.\sin Arq=2\rho.\sin C$. Also $Aq=AR.\frac{\sin A}{\sin B}=2R\sin A.\frac{\sin A}{\sin B}$, where R is the radius of the circle ABC. Therefore $\rho=R.\frac{a^2}{b^2}$.



To determine the length of AS.

In the triangle OAO', $OA=R$, $O'A=R.a^2/bc$, $\angle OAO'=\angle B-\angle C$, and $OO'^2=OA^2+O'A^2-2.OA.O'A.\cos\angle OAO'$.

But $\cos B\cos C=\frac{a^4-(b^2-c^2)^2}{4a^2bc}$, $\sin B\sin C=\frac{16\triangle^2}{4a^2bc}$, where $\triangle^2$ is the square of the area of the triangle ABC.

Hence, $\frac{OO'^2}{R^2}.b^2c^2=a^4+\ldots-b^2c^2-\ldots=\frac{a^2b^2c^2.e^2}{4R^2\sin^2\omega}$, where e is the eccentricity of the Brocard ellipse, and ω is the Brocard angle.

Hence $OO'=\frac{1}{2}.ae.\operatorname{cosec}\omega$. But $An.OO'=AO.AO'.\sin OAO'$.

Therefore $AS=2.An=\dfrac{2R\sin\omega}{e}.\dfrac{a(b^2-c^2)}{abc}$.

To determine the length of BS. Since AS is the radical axis of the two circles, we have $2.OO'$ times the perpendicular from B on AS equals the power of B for the circle Aqr, that is, equals $BA.Br$.

Hence $2.OO'.SB\sin C=c.BQ.\dfrac{\sin B}{\sin C}=b.R\sin(A-C)$. So that $SB=\dfrac{2R\sin\omega}{e}.\dfrac{b\ (a^2-c^2)}{abc}$. Similarly, $SB=\dfrac{2R\sin\omega}{e}.\dfrac{c\ (a^2-b^2)}{abc}$.

Hence the point S is unique, and its trilinear coordinates are

$$\frac{1}{a\ (b^2-c^2)},\quad \frac{1}{b\ (c^2-a^2)},\quad \frac{1}{c\ (a^2-b^2)}.$$

Therefore the circles described with radii $R.ac/b^2$ and $R.ab/c^2$ to touch BQ and CR, respectively in B and C, will also pass through S.

The feet of the perpendiculars from S on the sides of the triangles ABC and PQR are collinear. Let these feet be represented by x, y, z, x', y', z'. The diagram shows that y, z, and x' are on the Simson line of S for the triangle Aqr. Drawing the second circle, touching BQ in B, we see that z, x, and y' are collinear, and likewise for x, y, and z'.

Hence the six points are collinear.

The distances SP, SQ, SR may be readily found. Let $\angle SAP=\theta$.

Then $\sin\theta=\cos O'An=\dfrac{An}{O'A}=\dfrac{R\sin\omega}{e}.\dfrac{a\ (b^2-c^2)}{abc}.\dfrac{1}{R}.\dfrac{b}{a^2}\dfrac{c}{}=\dfrac{\sin\omega}{e}.\dfrac{b^2-c^2}{a^2}$.

Hence $SP=2RS\sin\theta=\dfrac{2R\sin\omega}{e}.\dfrac{b^2-c^2}{a^2}$.

If p be the perpendicular from S on the Simson line of S for ABC, and if SA, SB, SC make angles α, β, γ with the diameter through S, it is known that

$$p=2R.\cos\alpha.\cos\beta.\cos\gamma$$

$$=2R.\frac{SA}{2R}\frac{SB}{2R}.\frac{SC}{2R}=2R.\left(\frac{\sin\omega}{e}\right)^3.\frac{(b^2-c^2)\ (a^2-c^2)\ (a^2-b^2)}{a^2b^2c^2}.$$

And for the triangle PQR, $p'=2R.\left(\dfrac{\sin\omega}{e}\right)^3.\dfrac{(b^2-c^2)\ (a^2-c^2)\ (a^2-b^2)}{a^2b^2c^2}$.

Therefore, $p=p'$, as already demonstrated.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

300. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Stillwater, Okla.

If $\alpha, \beta, \gamma, \ldots$, are the roots of the equation $\sin mx - nx\cos mx = 0$, prove that $\tan^{-1}\frac{x}{\alpha} + \tan^{-1}\frac{x}{\beta} + \ldots + \tan^{-1}\frac{x}{\nu} = 0$.

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

This is problem 16, p. 328, Loney's *Trigonometry*. The problem is not clearly stated. In a letter December 2, 1908, from the author, he says he cannot now recall what he had in mind when he proposed the problem. As the problem reads, one would infer that if any ν roots of the given equation be taken and arcs formed whose tangents are any number, x, divided by these roots, the sum of these arcs is zero. But this is manifestly incorrect.

There is one way in which the statement is correct, viz: Suppose $\sin mx$ and $\cos mx$ be leveloped in series. The given equation then becomes

$$(m-n)x + m^2\left(\frac{n}{2\,!} - \frac{m}{3\,!}\right)x^3 - m^4\left(\frac{n}{4\,!} - \frac{m}{5\,!}\right)x^5 + \ldots \textit{ ad infinitum} = 0 \ldots (1).$$

One root of this equation is 0. Dividing the equation by x, we obtain

$$(m-n) + m^2\left(\frac{n}{2\,!} - \frac{m}{3\,!}\right)x^2 - m^4\left(\frac{n}{4\,!} - \frac{m}{5\,!}\right)x^4 + \ldots = 0 \ldots (2).$$

The roots of this equation, of which there are an infinite number, enter in pairs with opposite signs. Thus if $\alpha, \beta, \gamma, \ldots$ are roots, so are $-\alpha, -\beta, -\gamma, \ldots$, since the left hand member is an even function of x. Then, if the root, 0, is excluded, and if no arc exceeds, in absolute value, π radians, we have

$$\left.\tan^{-1}\frac{x}{\alpha} + \tan^{-1}\frac{x}{-\alpha} + \tan^{-1}\frac{x}{\beta} + \tan^{-1}\frac{x}{-\beta} + \ldots + \tan^{-1}\frac{x}{\nu} + \tan^{-1}\frac{x}{-\nu}\right]_{\nu=\infty} = 0.$$

The statement is also true if we take any $\nu/2$ pairs of roots.

If ν is increased indefinitely so that all roots except 0 are included, the statement, that the sum of the arcs $= n'\pi$, may be shown to be true, as follows: We have

$$\tan\left[\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{\beta}+\tan^{-1}\frac{x}{\gamma}+\ldots+\tan^{-1}\frac{x}{\nu}\right].$$

$$=\frac{x\,\Sigma\frac{1}{a}-x^3\,\Sigma\frac{1}{a\,\beta\,\gamma}+x^5\,\Sigma\frac{1}{a\beta\gamma\delta\varepsilon}-\ldots}{1-x^2\,\Sigma\frac{1}{a\,\beta}+x^4\,\Sigma\frac{1}{a\,\beta\,\gamma\,\delta}-\ldots}=0\ldots(3);\text{ for}$$

$$\Sigma\frac{1}{a}\equiv\frac{1}{a}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\ldots=\frac{\beta\gamma\delta\varepsilon\ldots+a\gamma\delta\epsilon\ldots+a\beta\delta\epsilon\ldots+}{a\beta\gamma\delta\varepsilon\ldots}.$$

The numerator of this fraction is the sum of the roots taken one less than all at a time, and is therefore equal to the coefficient of x which in equation (2) is zero, and the denominator is the product of the roots which in (2) is the known term, $m-n$. Hence, $\Sigma\frac{1}{a}=0$. Similarly, $\Sigma\frac{1}{a\,\beta\,\gamma}$ is a fraction whose numerator is the sum of the products of the roots taken three less than all at a time, and is therefore the coefficient of x^3 in (2), which is 0. Hence, $\Sigma\frac{1}{a\,\beta\,\gamma}=0$. Similarly, for the other terms of the numerator of (3). From similar considerations, $\Sigma\frac{1}{a\,\beta}$, $\Sigma\frac{1}{a\,\beta\,\gamma\,\delta}$, $\ldots\neq 0$.

Hence, $\tan\Sigma\left(\tan^{-1}\frac{x}{a}\right)=0$. Hence, $\Sigma\tan^{-1}\frac{x}{a}=n'\pi$ where n' is any integral positive or negative number.

Also solved by G. B. M. Zerr, and V. M. Spunar.

301. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

A is at Philadelphia, B at Chicago. A's personal equation is e; B's is E. When a star crosses A's meridian at time $t_1=8$ hours, 33 minutes, 24 seconds, he presses a button, telegraphing the fact to B, who receives it at time $t_2=7$ hours, 43 minutes, 23 seconds. When it crosses B's meridian at time $T_2=8$ hours, 33 minutes, 10 seconds, he telegraphs A, who receives it at time $T_1=9$ hours, 23 minutes, 11 seconds. They now exchange places, and on the second day following, B observes the transit at time $t'_1=8$ hours, 33 minutes, 26 seconds, and A gets the information at Chicago at time $t'_2=7$ hours, 43 minutes, 25 seconds. It crosses A's meridian at time $T'_2=8$ hours, 33 minutes, 12 seconds, and B gets the information at time $T'_1=9$ hours, 23 minutes, 13 seconds. Find the difference of longitude between Philadelphia and Chicago.

Solution by the PROPOSER.

The true times of the two transits at Philadelphia are t_1+e and T_1.

Hence, difference of time between Philadelphia and Chicago is $D=T_1+\ldots-t_1-e\ldots(1)$.

The true times of the two transits at Chicago are T_2+E and t_2.

Hence. $D=T_2+E-t_2\ldots(2)$.

(1)+(2) gives $2D=T_1+T_2-t_1-t_2+E-e\ldots(3)$.

Similarly, when A and B have changed places, $2D=T'_1+T'_2-t'_1-t'_2-E+e\ldots(4)$.

$\frac{1}{4}(3)+(4)$ gives $D=\frac{1}{4}(T_1+T_2+T'_1+T'_2-t_1-t_2-t'_1-t'_2)$. Substituting the values of the T's and t's, we have

$D=\frac{1}{4}$(3 hours, 19 minutes, 8 seconds) =49 minutes, 47 seconds, and $L=15D=12°\ 26'\ 45''$, the difference of longitude.

Also solved by V. M. Spunar.

GEOMETRY.

335. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Determine analytically, the point where three lines in a plane appear of equal length.

Solution by the PROPOSER.

In the most general solution we can take the rectangular axes of coordinates so that the axis of abscissas agrees, in direction, with one of the lines. Let this line be given by (0, 0); $(a, 0)$; the second and third lines are then represented by the coordinates (b, c); (d, f), and (h, k); (m, n), respectively. Also let (x, y) be the point where they appear equal. The tangent of the angle formed by the lines through (x, y); (b, c) and (x, y); (d, f) is

$$\frac{\frac{c-y}{b-x}-\frac{f-y}{d-x}}{1+\frac{(c-y)}{(b-x)}\frac{(f-y)}{(d-x)}}=\frac{cd-bf+(f-c)x+(b-d)y}{bd+cf-(b+d)x-(c+f)y+x^2+y^2}.$$

With similar expressions for the other lines.

$$\therefore \frac{ay}{ax-x^2-y^2}=\frac{cd-bf+(f-c)x+(b-d)y}{bd+cf-(b+d)x-(c+f)y+x^2+y^2}\ldots(1).$$

$$\frac{ay}{ax-x^2-y^2}=\frac{km-hn+(n-k)x+(h-m)y}{hm+kn-(h+m)x-(k+n)y+x^2+y^2}\ldots(2).$$

$$\begin{aligned}\therefore (f-c)x^3&+(cd-bf+ac-af+by-dy+ay)x^2\\&+(abf-acd-2aby+fy^2-cy^2)x+(abd+acf)y\\&+(cd-bf-ac-af)y^2+(a+b-d)y^3=0\ldots(3).\end{aligned}$$

$$(n-k)x^3+(km-hn+ak-an+hy-my+ay)x^2$$
$$+(ahn-akm-2ahy+ny^2-ky^2)x+(ahm+akn)y$$
$$+(km-hn-ak-an)y^2+(a+h-m)y^3=0\ldots(4).$$

(3) and (4) are the equations to determine x, y.

I. Let $c=f=k=n=0$. Then the three lines lie in the same straight line.

(3) becomes $(a+b-d)(x^2+y^2)=2abx-abd\ldots(5)$.

(4) becomes $(a+h-m)(x^2+y^2)=2ahx-ahm\ldots(6)$.

$$\therefore x=\frac{m(ah+bh+bd)-d(hm+ab+bd)}{2(ah-ab-dh+bm)}.$$

$$y=\pm\sqrt{\frac{2a(b-h)x+a(hm-bd)-(b-d-h+m)x^2}{b-d-h+m}}.$$

Let $a=b$, $h=$d. Then the three lines form one consecutive straight line.

$$\therefore x=\frac{3adm-d^2m-a^2d-ad^2}{2(ad-a^2-d^2+am)}.$$

$$y=\pm\sqrt{\frac{2a(a-d)x+a(dm-ad)-(a-2d+m)x^2}{a-2d+m}}.$$

Let $d=2a$, $m=3a$. Then the lines are all of equal length.

$\therefore x=0/0$, and is indeterminate; $y=\infty$.

Hence there is no point at which three equal straight lines forming a continuous straight line will appear equal to each other.

II. $b=a$ and $c=0$ in (3) gives

$$(fx-2af+2ay-dy)(x^2+y^2)+a^2(fx-2xy+dy)=0\ldots(7).$$

$h=d$, $k=f$, $m=n=0$ in (4) gives

$$(fx-af-ay-dy)(x^2+y^2)+2ay(dx+fy)=0\ldots(8).$$

The three lines now are the sides of a triangle. The solution of these equations in general terms would be quite difficult, yet one can determine the values of x and y by the following method: Let u, v, w be the distances from (x, y) to $(0, 0)$; $(a, 0)$, (d, f), respectively. Then we have

$$x^2+y^2=w^2,\ (a-x)^2+y^2=v^2,\ (d-x)^2+(f-y)^2=w^2.$$

$$\therefore\ x=\frac{a^2+u^2-v^2}{2a},\quad y=\frac{(u^2-ad)(a-d)+dv^2+af^2-aw^2}{2af}.$$

Also, $u^2+uv+v^2=a^2$,
$u^2+uw+w^2=d^2+f^2$,
$v^2+vw+w^2=(a-d)^2+f^2$.

$$\therefore\ u=\frac{ad\pm\frac{1}{3}af\sqrt{3}}{\sqrt{(a^2+d^2+f^2-ad\pm af\sqrt{3})}},$$

$$v=\frac{a^2-ad\pm\frac{1}{3}af\sqrt{3}}{\sqrt{(a^2+d^2+f^2-ad\pm af\sqrt{3})}},$$

$$w=\frac{d^2+f^2-ad\pm\frac{1}{3}af\sqrt{3}}{\sqrt{(a^2+d^2+f^2-ad\pm af\sqrt{3})}},$$

The plus sign is applicable to our solution.

Let $d=\frac{1}{2}a$, and $f=\frac{1}{2}a\sqrt{3}$. Hence, $u=\frac{1}{3}a\sqrt{3}=v=w$. Thus the values of x and y are determined, and equations (7) and (8) are solved.

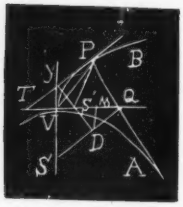
Hence $x=\dfrac{3ad^2+3af^2+3a^2d+a^2f\sqrt{3}+4adf\sqrt{3}}{6(a^2+d^2+f^2-ad+af\sqrt{3})}=\frac{1}{2}a$, when $d=\frac{1}{2}a$ and $f=\frac{1}{2}a\sqrt{3}$.

$$y=\frac{3a^2d\sqrt{3}-3ad^2\sqrt{3}+af^2\sqrt{3}+3a^2f}{6(a^2+d^2+f^2-ad+af\sqrt{3})}=\frac{1}{6}a\sqrt{3},\ \text{when } d=\tfrac{1}{2}a,\ f=\tfrac{1}{2}a\sqrt{3}.$$

Also solved by C. N. Schmall.

323. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

S, S' are the foci of two co-vertical parabolas A and B, the axes of which are at right angles. Draw the circle K on SS' as diameter. K is cut in D and E by a straight line parallel to the axis of A such that S' lies midway between it and that axis. Show that the lines $S'D$, $S'E$ are parallel to the two tangents to A which are normals to B.



II. Solution by (1) PROPOSER, and (2) R. F. DAVIS, M. A.

I°. The axis of B is the tangent at the common vertex V to A. Hence if PQ be normal at P to B, meeting the axis of B in Q, and if PQ is at the same time a tangent to A, it necessarily follows that SQ is perpendicular to PQ and therefore parallel to TP (the tangent at P to B). Or,

II°. $S'T=S'P=S'Q$ (letters as above). Draw $YS'D$ perpendicular to the parallels TP, SQ, and DM perpendicular to the axis of B (Y, of course, being on the tangent at V to B). Then $S'M=VS'$, and D is on the circle, SS' diameter, since $\angle SDS'=90°$.

III. Solution by the PROPOSER.

The foci S, S' of (A), $y^2=4ax$, and (B) $x^2=4by$, are, respectively, $(a, 0)$, $(0, b)$, and the circle in question has as its equation $x^2+y^2-ax-by=0$.

Since the common vertex V is $(0, 0)$ and S' is $(0, b)$, the point M in which DE cuts the axis of B is 0, $2b$, and the line DE is $y=2b$.

This cuts the circle where $x^2-ax+2b^2=0$...(1), giving D and E. If the roots of this be x_1 and x_2, D, E are $(x_1, 2b)$, $(x_2, 2b)$, and $S'D$, $S'E$ have respectively, for their equations,

$$bx-x_1y+x_1b=0 \text{ and } bx-x_2y+x_2b=0.$$

The tangent at $(at^2, 2at)$ to A is $x-ty+at^2=0$.

The normal at $(2bt', at'^2)$ to B is $x+t'y-2bt'-bt'^3=0$.

These are the same line if $1=\dfrac{-t}{t'}=\dfrac{-at^2}{bt'(2+t'^2)}$.

$\therefore t=-t'$, and rejecting the values $t=t'=0$ we have $t^2b-at+2b=0$...(2).

The roots of (1) and (2) are the same, and the property is proved.

Perhaps Prof. Zerr will reconsider the solution he offers.

CALCULUS.

263. Proposed by V. M. SPUNAR, M. S., C. E., East Pittsburg, Pa.

Find a point such that the sum of the squares of its distances from n given points shall be a minimum, and prove that the value so found is $1/n$th part of the sum of the squares of the mutual distances between the n points, taken two and two.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

This problem is proposed and solved in Williamson's *Differential Calculus*, 9th edition, page 192, Art. 157.

Taking one of the points as origin, and the axes rectangular, let (x, y) be the coordinates of the required point. Let (a_1, b_1); (a_2, b_2); (a_3, b_3); ..., (a_{n-1}, b_{n-1}) be the coordinates of the other $(n-1)$ points. Then

$$x^2+y^2+(x-a_1)^2+(y-b_1)^2+(x-a_2)^2+(y-b_2)^2+\ldots$$
$$+(x-a_{n-1})^2+(y-b_{n-1})^2=u=\text{minimum, or}$$

$$nx^2+ny^2-2(a_1+a_2+\ldots+a_{n-1})x-2(b_1+b_2+\ldots+b_{n-1})y$$
$$+a_1^2+a_2^2+\ldots+a^2{}_{n-1}+b_1^2+b_2^2+\ldots+b^2{}_{n-1}=u=\text{minimum.}$$

$$\therefore (nx-a_1-a_2-\ldots-a_{n-1})dx+(ny-b_1-b_2-\ldots-b_{n-1})dy=0.$$

$$\therefore x=\frac{a_1+a_2+\ldots+a_{n-1}}{n},\quad y=\frac{b_1+b_2+\ldots+b_{n-1}}{n}.$$

The point is, therefore, the center of mean position of the n points as stated.

Also solved by J. Scheffer.

264. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

The join of the center of curvature of a curve to the origin is at α to the initial line. Prove that with the usual notation:

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2+\left(\frac{d^2p}{d\psi^2}\right)^2\right]=\frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

No solution of this problem has been received.

265. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find two curves which possess the property that the tangents TP and TQ to the inner one always make equal angles with the tangent TT' to the outer.

No solution of this problem has been received.

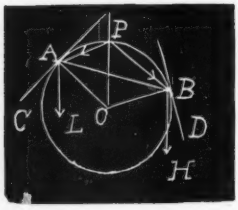
MECHANICS.

219. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

A rod length $a\sqrt{3}$, weight W, has at each end a smooth ring which can slide on a vertical circle radius r. Each ring is attached by an elastic string (natural lengths a, b; moduli μa, μb) to the highest point of the circle. Find the inclination of the rod to the horizon in a position of equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In what follows we regard the strings as having no weight, and also that both strings are in tension from the weight of the rod and rings, and the rod is above the center of the circle. Let $AB=a\sqrt{3}$=rod; AP=string a; BP=string b; O, the center of the circle, radius $AO=r$. Draw AK perpendicular to PO. Let m= weight of each ring; $\angle AOB=\beta=2\sin^{-1}(a\sqrt{3}/2r)$; $\angle APB=\pi-\frac{1}{2}\beta$; $\angle KAB=\theta$=angle AB makes with the horizon; $\angle PAB=\phi$; $\angle PBA=\frac{1}{2}\beta-\phi$; T=tension of string AP; T'=tension of string PB. When in equilibrium, W and the components of T, T' tangent at A, B meet in a point. $AP=a(1+T/\mu a)=(\mu a+T)/\mu$, $BP=(\mu b+T')/\mu$, $T=(\frac{1}{2}W+m)\sin(\phi-\theta)$. Let $(\frac{1}{2}W+m)=Q$.



$\therefore T=Q\sin(\phi-\theta)$, $T'=Q\sin(\theta+\frac{1}{2}\beta-\phi)$, $AP/PB=(\mu a+T)/(\mu b+T)=\sin(\frac{1}{2}\beta-\phi)/\sin\phi$... (1).

$3\mu^2a^2=(\mu a+T)^2+(\mu b+T')^2+2(\mu a+T)(\mu b+T')\cos\frac{1}{2}\beta$... (2).

The values of T and T' in (1) and (2) give

$$[\mu a+Q\sin(\phi-\theta)]\sin\phi=[\mu b+Q\sin(\theta+\tfrac{1}{2}\beta-\phi)]\sin(\tfrac{1}{2}\beta-\phi)\text{...}(3).$$

$$3\mu^2a^2=[\mu a+Q\sin(\phi-\theta)]^2+[\mu b+Q\sin(\theta+\tfrac{1}{2}\beta-\phi)]^2$$
$$+2[\mu a+Q\sin(\phi-\theta)][\mu b+Q\sin(\theta+\tfrac{1}{2}\beta-\phi)]\cos\tfrac{1}{2}\beta\ldots(4).$$

Eliminating ϕ between (3) and (4) gives an equation to determine θ.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

152. Proposed by H. S. VANDIVER, Bala, Pa.

When p is a prime of the form $5n\pm1$ then there is a positive integer a such that $a^2\equiv5\pmod p$. Show that $\left(\frac{a\pm1}{p}\right)=\pm\left(\frac{-2a}{p}\right)$, according as p is of the form $5n+1$ or $5n-1$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A general expression involving all cases is, apparently, not easily deduced. The following cases hold.

(1) Let $n=2$. Then $p=5n+1=11$. $a=4$, $a=p-4=7$, $a=2p-7=15$, $a=3p-15=18$, etc.

(2) Let $n=4$. Then $p=5n-1=19$, $a=9$, $a=p-9=10$, $a=2p-10=28$, $a=3p-28=29$, etc.

(3) Let $n=6$. Then $p=5n+1=31$, $a=6$, $a=p-6=25$, $a=2p-25=37$, $a=3p-37=56$, etc.

(4) Let $n=8$. Then $p=5n+1=41$, $a=13$, $a=p-13=28$, $a=2p-28=54$, $a=3p-54=69$, etc.

(5) Let $n=12$. Then (b) $p=5n+1=61$; (c) $p=5n-1=59$.

(b) $a=26$, $a=p-26=35$, $a=2p-35=87$, $a=3p-87=96$, etc.

(c) $a=8$, $a=p-8=51$, $a=2p-51=67$, $a=3p-67=110$, etc.

For every value of n that makes $5n\pm1$ a prime, we can find values for a satisfying the condition. It is also easy to see that a can have an infinitude of values for each case.

In (1), (3), (4), (5)(b), $(a+1)^{\frac{1}{2}(p-1)}=(-2a)^{\frac{1}{2}(p-1)}\equiv-1\pmod p$.

In (2), (5)(c), $(a-1)^{\frac{1}{2}(p-1)}=-(-2a)^{\frac{1}{2}(p-1)}\equiv1\pmod p$.

$\therefore\left(\frac{a\pm1}{p}\right)=\pm\left(\frac{-2a}{p}\right)$, in the cases examined, according as $p=5n\pm1$.

A general solution is desired. ED. F.

152. Proposed by H. S. VANDIVER, Bala, Pa.

Prove geometrically:

$\sum_{n=1}^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]=\frac{p-3}{4}\cdot\frac{p-1}{2}-\sum_{n=1}^{\frac{1}{2}(p-4}\left[\sqrt{np}\right]$, where $p\equiv3\pmod4$ and $\left[\frac{k}{p}\right]$ represents the greatest integer in k/p.

Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In the second term of the second member of the equation, $\frac{1}{4}(p-4)$ should be $\frac{1}{4}(p-3)$. The equation is generally but not universally true as is shown by induction in what follows.

Let $p=4m+3$, then $\frac{1}{2}(p-1)=2m+1$, $\frac{1}{4}(p-3)=m$,

$$\sum_{n=1}^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]=\frac{p-3}{4}\cdot\frac{p-1}{2}-\sum_{n=1}^{\frac{1}{4}(p-3)}[\sqrt{(np)},$$

or as follows:

$$\begin{aligned} &A= B-C,\\ m=2,\quad &3= 10-7,\\ m=3,\quad &7= 21-14,\\ m=4,\quad &11= 36-25,\\ m=5,\quad &18= 55-37,\\ m=7,\quad &34=105-71. \end{aligned}$$

If one of the $[n^2/p]$ is an exact quotient, and hence one of the $[\sqrt{(np)}]$ rational, the equation is $A=1+B-C$.

$$m=6,\ p=27,\ [9^2/p]=3,\ \sqrt{(3\times27)}=9,$$
$$m=15,\ p=63,\ 21^2/p=7,\ \sqrt{(7\times63)}=21.$$

$\therefore A=1+B-C$, $m=6\ldots25=1+78-54$, $m=15\ldots153=1+465-313$.

If two of the $[n^2/p]$ are exact quotients, and hence two of the $[\sqrt{(np)}]$ rational, the equation becomes $A=2+B-C$.

$$m=18,\ p=75,\ 15^2/p=3,\ \sqrt{(3\times p)}=15,\ 30^2/p=12,\ \sqrt{(12\times p)}=30.$$

$\therefore A=2+B-C$ becomes $219=2+666-449$ for $m=18$. $A=t+B-C$ is the true universal equation.

The geometric proof in this solution is wanting. Who can produce it? ED. F.

155. Proposed by PROF. R. D. CARMICHAEL, Anniston, Alabama.

If p and q are primes and m and n are any integers, find the cases in which the equation $p^m-q^n=1$ may be satisfied.

Remarks by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Some values, found by inspection, are given in the following table:

p	q	m	n
3	2	1	1
3	2	2	3
2	1	1	1
2	31	5	1
2	127	7	1

ERRATUM. In 156 for e^2 read e^3.

AVERAGE AND PROBABILITY.

196. Proposed by R. D. CARMICHAEL, Anniston, Ala.

A circle is inscribed in a square. Find the chance that the distance between two points within the square and without the circle shall not exceed a side of the square.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $2a$=side of square; (x, y), (u, v) the coordinates of the points; $\sqrt{(a^2-x^2)}=y_1$, $\sqrt{(a^2-u^2)}=v_1$. If both points are in the same corner, the distance between them is always less than $2a$. If both points are situated one each in opposite corners, the distance between them is always greater than $2a$. If both points are placed one each in adjacent corners, we have $\sqrt{[(x-u)^2+(y+v)^2]}=4a^2$, for the greatest distance between the points and $\sqrt{\{(x-u)^2+[\sqrt{(a^2-x^2)}+\sqrt{(a^2-u^2)}]^2\}}=\sqrt{[(x-u)^2+(y+v)^2]}$, for the least distance between the points.

$\therefore v_2=\sqrt{[4a^2-(x-u)^2]}-y$, $v_3=\sqrt{(a^2-x^2)}+\sqrt{(a^2-u^2)}-y$.

$$\therefore p=\frac{\int_0^a\int_0^a\left[\int_{y_1}^a\int_{v_1}^a dydv+\int_{y_1}^a\int_{v_1}^{v_2}dy\,dv\right]dxdu+\int_0^a\int_0^x\int_{y_1}^a\int_{v_2}^{v_2}dxdudydv}{3\int_0^a\int_0^a\int_{y_1}^a\int_{v_1}^a dx\,dy\,du\,dv}$$

$$=\frac{a^4(4-\pi)^2+0+.16\int_0^a\int_0^x\int_{y_1}^a\int_{v_3}^{v_2}dx\,du\,dy\,dv}{3a^4(4-\pi)^2}=\tfrac{1}{3}+\frac{1.6}{3a^4(4-\pi)^2}.M.$$

$$M=\int_0^a\int_0^x\int_{y_1}^a\{\sqrt{[4a^2-(x-u)^2]}-\sqrt{[a^2-x^2]}-\sqrt{[a^2-u^2]}\}dx\,du\,dy$$

$$=\int_0^a\int_0^x[a-\sqrt{(a^2-x^2)}]\{\sqrt{[4a^2-(x-u)^2]}-\sqrt{[a^2-x^2]}-\sqrt{[a^2-u^2]}\}dx\,du$$

$$=\int_0^a [a-\sqrt{(a^2-x^2)}]\left(\frac{x}{2}\sqrt{[4a^2-x^2]}-\tfrac{3}{2}x\sqrt{[a^2-x^2]}+2a^2\sin^{-1}\frac{x}{2a}\right.$$

$$\left.-\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)dx=a^4\int_0^{\frac{1}{2}\pi}(1-\cos\theta)\,[\tfrac{1}{2}\sin\theta\sqrt{(4-\sin^2\theta)}-\tfrac{3}{2}\sin\theta\cos\theta$$

$$+2\sin^{-1}(\tfrac{1}{2}\sin\theta)-\tfrac{1}{2}\theta]\cos\theta\,d\theta=a^4(\tfrac{3}{2}\sqrt{3}-\tfrac{9}{12}+\frac{\pi^2}{32}+\frac{\pi}{12})$$

$$-a^4\int_0^{\frac{1}{2}\pi}[\tfrac{1}{2}\sin\theta\sqrt{)}4-\sin^2\theta)+2\sin^{-1}(\tfrac{1}{2}\sin\theta)]\cos^2\theta\,d\theta$$

$$=a^4(\tfrac{3}{2}\sqrt{3}-\tfrac{9}{12}+\frac{\pi^2}{32}+\frac{\pi}{12})-a^4\int_0^{\frac{1}{2}\pi}(2\sin\theta-\tfrac{1}{8}\sin^2\theta+\frac{15}{2^7}\sin^3\theta-\frac{1}{2^{10}}\sin^4\theta$$

$$+\frac{763}{2^{15}}\sin^5\theta-\frac{7}{2^{18}}\sin^6\theta+\frac{10219}{2^{22}}\sin^7\theta-\ldots)\cos^2\theta\,d\theta.$$

$$\therefore M=a^4(\tfrac{3}{2}\sqrt{3}-\tfrac{9}{12}+\frac{\pi^2}{32}+\frac{\pi}{12}-0.6595495)=0.0412517a^4.$$

$$\frac{16M}{3a^4(4-\pi)^2}=\frac{0.6600272}{2.2108850}=\tfrac{3}{10},\text{ nearly, }=.299.\quad\therefore p=\tfrac{1}{8}+\tfrac{3}{10}=\tfrac{19}{30},\text{ nearly.}$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

208. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Find the conditions that the roots of $x^2+px+q=0$ may not lie between -1 and $+1$.

209. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve, $bx^2+cy^2+az^2=ba^2+cb^2+ac^2$,
$cx^2+ay^2+bz^2=ab^2+bc^2+ca^2$,
$xyz=abc$.

210. Proposed by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Simplify, $\log[\sqrt[8]{(137)}\sqrt[7]{(56)}\div\sqrt[10]{(187)}\sqrt[4]{(75)}]$.

GEOMETRY.

341. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

Given $\rho=\cos(m/n)\theta$, where m and n are integers without a common factor. Deduce rules for finding by inspection:

(1) The angle between the beginning and end of any loop of this curve; (2) The number of distinct loops. [A loop is a portion of the curve between consecutive zero radii vectores.]

CALCULUS.

267. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

A point within an ellipse, upon a normal making an angle λ with the major axis, is arbitrarily chosen. With this point as pole, and the line through it parallel to the major axis as polar axis, the equation of the ellipse is, $A\cos^4\theta+B\cos^3\theta+C\cos^2\theta+D\cos\theta+E=0$, where the coefficients are functions of λ, of the radius vector ρ, and of the distance along the normal to the pole, ρ_1. Evidently for $\rho=\rho_1$, a solution is $\cos\theta=\cos\lambda$. Required the multiplicity of this solution for any values of ρ_1, $[\lambda\neq 0,\ \rho_1\neq 0]$.

268. Proopsed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Determine $\phi(y)$, independent of u, so that the equation $\int_0^u (u-y)^{(2p-1)/2}\phi(y)dy=u^m$ is satisfied, p and m being positive integers and $m>p$. Do you notice properties of special interest for any special cases?

MECHANICS.

223. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A sphere, radius $r=\frac{1}{3}$ inches, density $\delta=11.38$, falls from a height $h=$ 500 feet, into a lake depth $l=40$ feet. Find time of falling to surface of lake, time of falling from surface of lake to bottom, and total time of falling. Also the velocity at the bottom.

224. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A steel clock spring $w=\frac{7}{8}$ inch wide, $t=\frac{1}{32}$ inch thick, is wound around an axle $d=\frac{1}{4}$ inch in diameter. Find the greatest available moment for running the clock, using a factor of safety $f=6$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

159. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Show that if the equation $y^3=2x^2-1$ be possible in integers, $y=24n^2-1$, or $2n^2-1$, and find three solutions.

AVERAGE AND PROBABILITY.

200. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

A line $AB=l$ is extended to P making $BP=p$. If a point D is taken at random in BP, what is the mean value of $AD.DP$?

201. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A random straight line is drawn across a circle and another through a given point on the circumference. Find the chance that they intersect within the circle.

202. Proposed by F. P. MATZ, Ph. D., Reading, Pa.

If three chords are drawn at random in a circle, what is the chance the center of the circle is enclosed by the three chords, and what is the mean area of this enclosing triangle?

203. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Find the average length of a hole at random through a given cone.

MISCELLANEOUS.

180. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Show that $\sum_{n=1}^{n=\infty}(-1)\frac{1}{(2n-1)(2^{2n}-1)} = \sum \tan^{-1}\frac{3.2^n}{2^{2n+1}-1}$, the smallest value of each inverse function being taken.

NOTES AND NEWS.

Dr. Herbert G. Keppel, of Northwestern University, has been elected head of the department of mathematics of the University of Florida.

The next annual meeting of the American Mathematical Society will be held at Baltimore, Md., during the holidays in connection with the annual conference of the American Society for the Advancement of Science.

At the meeting of the Central Association of Scienc and Mathematics Teachers, held in Chicago, November 27 and 28, 1908, a committee was appointed to select material for concrete applications in the teaching of geometry. This is a move in the right direction, for without doubt geometry teaching has become too much of an exercise in abstract logic.

A movement is under way whereby the recently organized Federation of Associations of Science and Mathematics Teachers throughout the

country may undertake definite work in the matter of educational improvement in these lines. One important step contemplated is the appointment of a representative committee to prepare a national syllabus on geometry.

In Science, for October 23rd, appears an article by Professor G. A. Miller, of the University of Illinois, on the "Method of Appointment of University Professors in Foreign Countries." This topic will also be discussed by Professor Wilczynski, of the University of Illinois, at the meetof the Chicago Section of the American Mathematical Society to be held in Chicago, January 1 and 2, 1909.

At the recent meeting of the New York section of the Association of Teachhers of Mathematics in the Middle States and Maryland the general topic for discussion was:

"Modern Tendencies in the Teaching of Algebra, or An Evening with the Writers of Some of the Modern Text-books in Algebra."

The speakers were:

Fletcher Durell, Ph. D., Mathematical Master in the Lawrenceville School, joint author, with Edward R. Robbins, A. B., of the Durell and Robbins Series; Isaac J. Schwatt, Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania, joint author with Dr. G. E. Fisher, of the Fisher and Schwatt Series; N. J. Lennes, Ph. D., Instructor in Mathematics, Brown University, joint author with Dr. H. E. Slaught, of the Slaught and Lennes Series.

The discussions were led by the following:

Mr. Wm. E. Breckenridge, Stuyvesant High School, New York City; Mr. Merle L. Bishop, Boys' High School, Brooklyn; Mr. Oscar W. Anthony, Dewitt Clinton High School, New York City.

This issue of the MONTHLY was mailed February 1.

Dr. T. W. Wright, author of a good text-book on Mechanics, Professor of Mathematics in Union College, and a subscriber of the MONTHLY from the first, died last September.

Mr. A. H. Bell, of Litchfield, Illinois, a valued contributor and subscriber of the MONTHLY from its beginning, is still doing civil engineering work though he is past 78 years of age.

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CALCULUS. Gunther's *Integration by Trigonometric and Imaginary Substitution*, p. 24; Veblen and Lennes' *Introduction to Infinitesimal Analysis*, p. 116; Murray's *Differential and Integral Calculus*, p. 194; Johnson's *Elementary Treatise on the Differential and Integral Calculus*, p. 215.

FUNCTIONS. Hobson's *Theory of Functions of a Real Variable and Fourier's Series*, p. 94.

GEOMETRY. Nichol's *Analytical Geometry*, p. 215; Lyman's *Plane and Solid Geometry*, p. 216.

MATHEMATICS. Woods and Bailey's *Course in Mathematics*, p. 24; White's *Scrap-Book of Elementary Mathematics*, p. 70; Waterbury's *Vest-Pocket Handbook of Mathematics for Engineers*, p. 116; Fabry's *Traité de Mathématiques Généralés*, p. 215; Carus' *Foundations of Mathematics*, p. 216.

NUMBERS. Manning's *Irrational Numbers*, p. 23.

PHYSICS. Crew's *General Physics*, p. 193; Millikan and Mills' *Short Course in Electricity, Sound, and Light*, p. 194; Hoadley's *Elements of Physics*, p. 215.

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